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Bi-modal Gödel logic over [0,1]-valued Kripke frames

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Abstract

We consider the Gödel bi-modal logic determined by fuzzy Kripke models where both the propositions and the accessibility relation are infinitely valued over the standard Gödel algebra [0,1], and prove strong completeness of the Fischer Servi intuitionistic modal logic *IK* plus the prelinearity axiom with respect to this semantics. We axiomatize also the bi-modal analogues of classical T, $S4$ and $S5$, obtained by restricting to models over frames satisfying the [0,1]-valued versions of the structural properties which characterize these logics. As an application of the completeness theorems we obtain a representation theorem for bi-modal Gödel algebras.

Keywords: Gödel logic, modal logic, fuzzy logic, Kripke models, modal algebras, many-valued logics.

In a previous paper [6], we considered a semantics for Gödel modal logic based on fuzzy Kripke models where both the propositions and the accessibility relation take values in the standard Gödel algebra [0,1], we call these Gödel–Kripke models, and we provided strongly complete axiomatizations for the uni-modal fragments of this logic with respect to validity and semantic entailment from countable theories. The systems $\mathcal{G}_\Box$ and $\mathcal{G}_\Diamond$ axiomatizing the $\Box$-fragment and the $\Diamond$-fragment, respectively, are obtained by adding to Gödel–Dummett propositional calculus the following axiom schemes and inference rules:

$$\begin{array}{ll} \mathcal{G}_\Box\colon \Box(\varphi\to\psi)\to(\Box\varphi\to\Box\psi) & \qquad \mathcal{G}_\Diamond\colon \Diamond(\varphi\vee\psi)\to(\Diamond\varphi\vee\Diamond\psi) \\ \quad\ \neg\neg\Box\varphi\to\Box\neg\neg\varphi & \qquad\quad\ \Diamond\neg\neg\varphi\to\neg\neg\Diamond\varphi \\ \quad\ \text{From } \varphi,\ \text{infer } \Box\varphi & \qquad\quad\ \neg\Diamond\bot \\ & \qquad\quad\ \text{From } \varphi\to\psi,\ \text{infer } \Diamond\varphi\to\Diamond\psi. \end{array}$$

These two logics diverge substantially in their model theoretic properties. Thus, $\mathcal{G}_\Box$ does not have the finite model property while $\mathcal{G}_\Diamond$ does, and the first logic is characterized by models with $\{0,1\}$-valued accessibility relation (accessibility-crisp models) while the second one does not. Similar results were obtained for the uni-modal Gödel analogues of the classical modal logics T and $S4$ determined by Gödel–Kripke models over frames satisfying, respectively, the [0,1]-valued version of reflexivity, or reflexivity and transitivity. The axiomatization of the uni-modal Gödel analogues of $S5$ remains open.

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It is the main purpose of this article to show that the full bi-modal logic based on Gödel–Kripke models is axiomatized by the system $\mathcal{G}_{\Box\Diamond}$ that results by adding to the union of $\mathcal{G}_{\Box}$ and $\mathcal{G}_{\Diamond}$ the Fischer Servi's connecting axioms [14]:

$$\Diamond(\varphi \to \psi) \to (\Box\varphi \to \Diamond\psi)$$
$$(\Diamond\varphi \to \Box\psi) \to \Box(\varphi \to \psi),$$

and to extend this completeness result to the bi-modal Gödel analogues of classical T, $S4$, $S5$ and related systems.

We discuss briefly at the end of the article an embedding of our semantics into algebraic semantics for $\mathcal{G}_{\Box\Diamond}$ and its extensions, and utilize our completeness theorem to show a functional representation theorem for bi-modal Gödel algebras.

The many valued Kripke interpretation of bi-modal logic utilized in this article was proposed originally by Fitting [15], [16], with a complete Heyting algebra as algebra of truth values, and he gave a complete axiomatization assuming the algebra was finite and the language had constants for all the truth values. See also [21] and [11]. Bou, Esteva and Godo [5] have proposed utilizing this kind of interpretation for general algebras in the study of fuzzy modal logics. Our methods of proof do not seem to extend easily, however, to algebras distinct from the Gödel algebra [0,1], and we do not know any other completeness result for this type of semantics for a fixed algebra H, except Fitting's result quoted above and Metcalfe and Olivetti completeness of analytic Gentzen systems for $\mathcal{G}_{\Box}$ and $\mathcal{G}_{\Diamond}$ [22].

$\mathcal{G}_{\Box\Diamond}$ is equivalent to the system *IK*, proposed by Fischer Servi [14] as the intuitionistic counterpart of classical modal logic K, plus the prelinearity axiom: $(\varphi \to \psi) \vee (\psi \to \varphi)$. Similarly, the Gödel analogue of bi-modal S5 is equivalent to the system *MIPC* of Bull [4] and Prior [26] plus prelinearity.

IK and its extensions have been extensively studied, either by means of classical Kripke models for intuitionism equipped with extra relations commuting with the order to interpret the modal operators ([28], [25], [10], [29], [30], [17], [7], [9]), or by means of algebraic interpretations, specially in the case of *MIPC*, known to be complete for values in monadic Heyting algebras ([4], [24], [13], [1], [2]). A major result is that both logics enjoy the finite model property under these semantics. Clearly, $\mathcal{G}_{\Box\Diamond}$ and its modal extensions inherit similar semantics, but those interpretations do not have the standard character of Gödel–Kripke semantics relevant to fuzzy logic, and it does not seem possible to derivate our results from their properties. For example, the formula $\Box\neg\neg\theta \to \neg\neg\Box\theta$ has finite counter-models with respect to those semantics but not for Gödel–Kripke semantics.

1 Gödel–Kripke models

The language $\mathcal{L}_{\Box\Diamond}(Var)$ of propositional *bi-modal logic* is built from a set *Var* of propositional variables, connective symbols $\vee, \wedge, \to, \bot$ and the modal operator symbols $\Box$ and $\Diamond$. Other connectives are defined as usual: $\top := \varphi \to \varphi$, $\neg\varphi := \varphi \to \bot$, $\varphi \leftrightarrow \psi := (\varphi \to \psi) \wedge (\psi \to \varphi)$. We will write $\mathcal{L}_{\Box\Diamond}$ if the set *Var* is understood.

Recall that a *linear Heyting algebra*, or *Gödel algebra* in the fuzzy literature, is a Heyting algebra satisfying the identity $(x \Rightarrow y) \curlyvee (y \Rightarrow x) = 1$. The variety of these algebras is generated by the *standard Gödel algebra* $[0,1]$, the ordered interval with its unique Heyting algebra structure. Here, the symbols $\cdot$, $\Rightarrow$, $\curlyvee$, denote, respectively, the meet, residuum (implication), and join operations of Heyting algebras. For convenience, we take $\curlyvee$ as primitive although it is definable in Gödel algebras as $x \curlyvee y = ((x \Rightarrow y) \Rightarrow y)) \cdot ((y \Rightarrow x) \Rightarrow x))$.

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DEFINITION 1.1
A Gödel–Kripke model (GK-model) will be a structure $M=\langle W,S,e\rangle$ where W is a non-empty set of objects that we call worlds of M, and $S:W\times W\to[0,1]$, $e:W\times Var\to[0,1]$ are arbitrary functions. The pair $\langle W,S\rangle$ will be called a GK-frame.

The function $e:W\times Var\to[0,1]$ associates to each world x a valuation $e(x,-):Var\to[0,1]$ which extends to $e(x,-):\mathcal{L}_{\Box\Diamond}(Var)\to[0,1]$ by defining inductively on the construction of the formulas (we utilize the same symbol e to name the extension):

$e(x,\bot):=0$
$e(x,\varphi\wedge\psi):=e(x,\varphi)\cdot e(x,\psi)$
$e(x,\varphi\vee\psi):=e(x,\varphi)\curlyvee e(x,\psi)$
$e(x,\varphi\to\psi):=e(x,\varphi)\Rightarrow e(x,\psi)$
$e(x,\Box\varphi):=\inf_{y\in W}\{Sxy\Rightarrow e(y,\varphi)\}$
$e(x,\Diamond\varphi):=\sup_{y\in W}\{Sxy\cdot e(y,\varphi)\}$.

Truth, validity and entailment are defined for $\varphi\in\mathcal{L}_{\Box\Diamond}$, $T\subseteq\mathcal{L}_{\Box\Diamond}$ as follows:

φ is *true in M at x*, written $M\models_x\varphi$, if $e(x,\varphi)=1$.
φ is *valid in M*, written $M\models\varphi$, if $M\models_x\varphi$ at any world x of M.
φ is *GK-valid*, written $\models_{GK}\varphi$, if $M\models\varphi$ for any GK-model M.
$T\models_{GK}\varphi$ if and only if for any GK-model M and any world x in M:

$$M\models_x\theta \text{ for all } \theta\in T \text{ implies } M\models_x\varphi.$$

It is routine to verify that all axiom schemes corresponding to identities satisfied in $[0,1]$; that is, the laws of Gödel–Dummett logic, are GK-valid. In addition

PROPOSITION 1.1
The following schemes are GK-valid:

($K_\Box$) $\Box(\varphi\to\psi)\to(\Box\varphi\to\Box\psi)$
($K_\Diamond$) $\Diamond(\varphi\vee\psi)\to(\Diamond\varphi\vee\Diamond\psi)$
($F_\Diamond$) $\neg\Diamond\bot$
(FS1) $\Diamond(\varphi\to\psi)\to(\Box\varphi\to\Diamond\psi)$
(FS2) $(\Diamond\varphi\to\Box\psi)\to\Box(\varphi\to\psi)$.

PROOF. Let $M=\langle W,S,e\rangle$ be a GK-model. ($K_\Box$): By definition and properties of the residuum, $e(x,\Box(\varphi\to\psi))\cdot e(x,\Box\varphi)\le(Sxy\Rightarrow(e(y,\varphi)\Rightarrow e(y,\psi))\cdot(Sxy\Rightarrow e(y,\varphi))\le(Sxy\Rightarrow e(y,\psi))$ for any $y\in W$. Taking the meet over y in the last expression: $e(x,\Box(\varphi\to\psi))\cdot e(x,\Box\varphi)\le e(x,\Box\psi)$, hence $e(x,\Box(\varphi\to\psi))\le e(x,\Box\varphi\to\Box\psi)$. ($K_\Diamond$): By distributivity and properties of the join: $e(\Diamond(x,\varphi\vee\psi))=\sup_y\{Sxy\cdot(e(y,\varphi)\curlyvee e(y,\psi))\}=\sup_y\{Sxy\cdot e(y,\varphi)\}\curlyvee\sup_y\{Sxy\cdot e(y,\psi)\}$. ($F_\Diamond$): $e(x,\Diamond\bot)=\sup_y\{Sxy\cdot 0\}=0$. (FS1): $Sxy\cdot e(x,\Box\varphi)\cdot e(y,\varphi\to\psi)\le Sxy\cdot(Sxy\Rightarrow e(y,\varphi))\cdot(e(y,\varphi)\Rightarrow e(y,\psi))\le Sxy\cdot e(y,\psi))\le e(x,\Diamond\psi)$. Therefore, $Sxy\cdot e(y,\varphi\to\psi)\le(e(x,\Box\varphi)\Rightarrow e(x,\Diamond\psi))$, and taking the join over y in the left-hand side, we have $e(x,\Diamond(\varphi\to\psi))\le e(x,\Box\varphi\to\Diamond\psi)$. (FS2): $e(x,\Diamond\varphi\to\Box\psi)\le[Sxy\cdot e(y,\varphi)\Rightarrow(Sxy\Rightarrow e(y,\psi))]=[Sxy\cdot e(y,\varphi)\Rightarrow e(y,\psi)]=(Sxy\Rightarrow e(y,\varphi\to\psi))$. ■

REMARK. Utilizing any complete Heyting algebra H instead of $[0,1]$ in the above definitions, we obtain H-valued Kripke models (*HK-models*) and corresponding notions of HK-validity and entailment. Then the laws in Proposition 1.1 are HK-valid, as are the laws of the intermediate propositional logic determined by H.

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2 A bi-modal calculus

Let $\mathcal{G}$ be some axiomatic version of Gödel–Dummett propositional calculus; that is, Heyting calculus plus the axiom $(\varphi \to \psi) \vee (\psi \to \varphi)$, and let $\vdash_{\mathcal{G}}$ denote deduction in this logic. Let $\mathcal{L}(X)$ denote the set of formulas built by means of the connectives $\wedge, \to$, and $\bot$, from a given set X. For simplicity, the extension of a valuation $v : X \to [0,1]$ to $\mathcal{L}(X)$ according to the Heyting interpretation of the connectives will be denoted v also. It is well known that this system is complete for validity with respect to these valuations and the distinguished value 1. We will need the fact that it is actually sound and complete in the following stronger sense (see [6], Prop. 3.1):

PROPOSITION 2.1
(i) If $T \cup \{\varphi\} \subseteq \mathcal{L}(X)$, then $T \vdash_{\mathcal{G}} \varphi$ implies $\inf v(T) \leq v(\varphi)$ for any valuation $v : X \to [0,1]$. (ii) If T is countable, and $T \nvdash_{\mathcal{G}} \varphi_{i_1} \vee .. \vee \varphi_{i_1}$ for each finite subset of a countable family $\{\varphi_i\}_i$ there is a valuation $v : L \to [0,1]$ such that $v(\theta) = 1$ for all $\theta \in T$ and $v(\varphi_i) < 1$ for all i.

For an example that completeness for [0,1]-valued entailment cannot be extended to uncountable theories see Section 3 in [6] and also Proposition 3.1 below.

DEFINITION 2.1
$\mathcal{G}_{\Box\Diamond}$ is the deductive calculus obtained by adding to $\mathcal{G}$ the schemes of Proposition 1.1 and the inference rules:

(NR$_\Box$) *From* φ *infer* $\Box\varphi$
(RN$_\Diamond$) *From* $\varphi \to \psi$ *infer* $\Diamond\varphi \to \Diamond\psi$.

Proofs with assumptions are allowed with the restriction that NR$_\Box$ and RN$_\Diamond$ may be applied only when the premise is a theorem; $\vdash_{\mathcal{G}_{\Box\Diamond}}$ will denote deduction in this system.

The restriction on the application of the rules allows the Deduction Theorem that we will utilize freely without quoting it:

LEMMA 2.1
$T, \psi \vdash_{\mathcal{G}_{\Box\Diamond}} \varphi$ implies $T \vdash_{\mathcal{G}_{\Box\Diamond}} \psi \to \varphi$.

An alternative axiomatization of $\mathcal{G}_{\Box\Diamond}$ is obtained by replacing FS1 with the scheme

(P) $\Box(\varphi \to \psi) \to (\Diamond\varphi \to \Diamond\psi)$

and deleting the rule RN$_\Diamond$.

Indeed, $\vdash_{\mathcal{G}_{\Box\Diamond}} \Diamond\varphi \to \Diamond((\varphi \to \psi) \to \psi)$ $\vdash_{\mathcal{G}_{\Box\Diamond}} \Diamond\varphi \to (\Box(\varphi \to \psi) \to \Diamond\psi)$ $\vdash_{\mathcal{G}_{\Box\Diamond}} \Box(\varphi \to \psi) \to (\Diamond\varphi \to \Diamond\psi)$ by Heyting calculus, RN$_\Diamond$ and FS1. On the other hand, P + NR$_\Box$ deduce trivially RN$_\Diamond$, and FS1 is deduced from $(\mathcal{G}_{\Box\Diamond} \setminus \{\text{FS1}, \text{RN}_\Diamond\}) + \{\text{P}\}$ as follows: $\vdash \Box\varphi \to \Box((\varphi \to \psi) \to \psi)$ by Heyting calculus plus NR$_\Box$ and K$_\Box$; thus $\vdash \Box\varphi \to (\Diamond(\varphi \to \psi) \to \Diamond\psi)$ by P, and $\vdash \Diamond(\varphi \to \psi) \to (\Box\varphi \to \Diamond\psi)$ by Heyting calculus.

THEOREM 2.1 (Soundness)
$T \vdash_{\mathcal{G}_{\Box\Diamond}} \varphi$ implies $T \models_{GK} \varphi$.

PROOF. Clearly, the Modus Ponens rule preserves truth at every world of any GK-model M, and the rule $NR_\Box$ preserves validity (truth at all worlds) in any model since $M \models \varphi$ implies $M \models \Box\varphi$, trivially. Similarly, $M \models \varphi \to \psi$ implies $Sxy \cdot e(y, \varphi) \leq Sxy \cdot e(y, \psi)$ for all x, y, and thus $M \models \Diamond\varphi \to \Diamond\psi$. The rest follows from Proposition 1.1. ■

It is easy to provide counterexamples to the validity of $\neg\Box\neg\theta \to \Diamond\theta$ and $\neg\Diamond\neg\theta \to \Box\theta$; thus the modal operators are not interdefinable in $\mathcal{G}_{\Box\Diamond}$ in the classical way. In fact, they are not interdefinable

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in any manner. For example, the invalid formula $\Box\neg\neg\theta \to \neg\neg\Box\theta$ is not expressible in terms of $\Diamond$ alone because the $\Diamond$-fragment has the finite model property for GK-semantics with respect to the number of worlds, while this formula has no finite counterexample (cf. [6], Prop. 5.1).

The following are some theorems of $\mathcal{G}_{\Box\Diamond}$. The first one is an axiom in Fitting's systems in [15], the next two show the fact claimed in the introduction that $\mathcal{G}_{\Box\Diamond}$ is the union of $\mathcal{G}_{\Box}$ and $\mathcal{G}_{\Diamond}$ plus the Fischer Servi axioms, the fourth one will be useful in our completeness proof and is the only one depending on prelinearity.

T1. $\neg\Diamond\theta \leftrightarrow \Box\neg\theta$
T2. $\neg\neg\Box\theta \to \Box\neg\neg\theta$
T3. $\Diamond\neg\neg\varphi \to \neg\neg\Diamond\varphi$
T4. $(\Box\varphi \to \Diamond\psi) \vee \Box((\varphi \to \psi) \to \psi)$

To see this, write temporarily $\vdash$ for $\vdash_{\mathcal{G}_{\Box\Diamond}}$. (T1) $\neg\Diamond\theta \vdash (\Diamond\theta \to \Box\bot) \vdash \Box(\theta \to \bot)$ by Heyting calculus and FS2. Similarly, $\Diamond\theta \vdash \Diamond(\neg\theta \to \bot) \vdash \Box\neg\theta \to \Diamond\bot \vdash \neg\Box\neg\theta$ by Heyting calculus, RN$_\Diamond$, and FS2; thus, $\Box\neg\theta \vdash \neg\Diamond\theta$ by Heyting calculus. (T2) $(\Box\varphi \to \bot) \to \bot \vdash (\Box\varphi \to \Diamond\bot) \to \bot \vdash \Diamond(\varphi \to \bot) \to \Box\bot$ $\vdash \Box((\varphi \to \bot) \to \bot)$ by F$_\Diamond$, FS2 and FS1. (T3) $\Diamond(\neg\varphi \to \bot) \vdash (\Box\neg\varphi \to \Diamond\bot) \vdash (\neg\Diamond\varphi \to \bot)$ by FS1, T1 and F$_\Diamond$. (T4) By prelinearity: $\vdash (\Box\varphi \to \Diamond(\varphi \to \psi)) \vee (\Diamond(\varphi \to \psi) \to \Box\varphi)$, but $\Box\varphi \to \Diamond(\varphi \to \psi) \vdash$ $\Box\varphi \to (\Box\varphi \to \Diamond\psi) \vdash \Box\varphi \to \Diamond\psi$ by FS1; moreover, $\Diamond(\varphi \to \psi) \to \Box\varphi \vdash \Box((\varphi \to \psi) \to \varphi) \vdash \Box((\varphi \to \psi) \to \psi)$ by FS2, Heyting calculus and RN$_\Box$.

To prove the completeness of $\vdash_{\mathcal{G}_{\Box\Diamond}}$, we will utilize the following convenient reduction of $\mathcal{G}_{\Box\Diamond}$ to pure Gödel calculus:

LEMMA 2.2
Let $Th\mathcal{G}_{\Box\Diamond}$ be the set of theorems of $\mathcal{G}_{\Box\Diamond}$ with no assumptions, then for any theory T and formula φ in $\mathcal{L}_{\Box\Diamond}: T \vdash_{\mathcal{G}_{\Box\Diamond}} \varphi$ if and only if $T \cup Th\mathcal{G}_{\Box\Diamond} \vdash_{\mathcal{G}} \varphi$.

PROOF. The rules NR$_\Box$, RN$_\Diamond$ are applied only to formulas in $Th\mathcal{G}_{\Box\Diamond}$, and this set is closed under those rules. ∎

REMARK. $\mathcal{G}_{\Box\Diamond}$ is essentially the Fischer Servi system *IK* ([14], [28]) plus the prelinearity axiom. Moreover, $T \vdash_{IK} \varphi$ implies $T \models_{HK} \varphi$ for any complete H. This provides a new interpretation of *IK* under which one would expect this logic to be complete.

3 Completeness

In this section, we prove strong completeness of $\mathcal{G}_{\Box\Diamond}$ with respect to entailment from countable theories in Gödel-Kripke semantics.

We will obtain a finer result for theories $T \subseteq \mathcal{L}_{\Box\Diamond}$ closed under the rule NR$_\Box$ ($T \vdash_{\mathcal{G}_{\Box\Diamond}} \theta$ implies $T \vdash_{\mathcal{G}_{\Box\Diamond}} \Box\theta$). Call these theories *normal*. It follows from the observation on an alternative axiomatization in the previous section that a normal theory is also closed under the rule RN$_\Diamond$. Clearly, the empty theory is normal.

Our strategy is to show first completeness for entailment from finite theories (weak completeness), and utilize a first order compactness argument to lift this to countable theories. To achieve the first goal, we define for each normal theory Σ and finite *fragment* $F \subseteq \mathcal{L}_{\Box\Diamond}$ (a subset closed under subformulas and containing the formula $\bot$) a canonical model $M_{\Sigma,F}$ in which $\Sigma \cap F$ will be valid.

Let $X := \{\Box\theta, \Diamond\theta : \theta \in \mathcal{L}_{\Box\Diamond}\}$ be the set of formulas in $\mathcal{L}_{\Box\Diamond}$ beginning with a modal operator; then $\mathcal{L}_{\Box\Diamond}(Var) = \mathcal{L}(Var \cup X)$. That is, any formula in $\mathcal{L}_{\Box\Diamond}(Var)$ may be seen as Heyting calculus formula

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built from the set of propositional variables $Var \cup X$. The *canonical model* $M_{\Sigma,F} = (W^\Sigma, S^F, e^F)$ is defined as follows:

- W^Σ is the set of valuations $v \in [0,1]^{Var \cup X}$ such that $v(\Sigma \cup Th\mathcal{G}_{\Box\Diamond}) = 1$, where $\Sigma \cup + Th\mathcal{G}_{\Box\Diamond}$ is considered a subset of $\mathcal{L}(Var \cup X)$.
- $S^F vw = \inf_{\psi \in F}\{(v(\Box\psi) \to w(\psi)) \cdot (w(\psi) \to v(\Diamond\psi))\}$.
- $e^F(v,p) = v(p)$ for any $p \in Var$.

Weak completeness will follow from the following lemma which has a rather involved proof.

LEMMA 3.1
$e^F(v,\varphi) = v(\varphi)$ for any $\varphi \in F$ and any $v \in W^\Sigma$.

PROOF. For simplicity, write W for W^Σ. We prove the identity by induction on the complexity of the formulas in F, considered now elements of $\mathcal{L}_{\Box\Diamond}(Var)$. For $\bot$ and the propositional variables in F the equation holds by definition. The only non-trivial inductive steps are: $e^F(v,\Box\varphi) = v(\Box\varphi)$ and $e^F(v,\Diamond\varphi) = v(\Diamond\varphi)$ for $\Box\varphi, \Diamond\varphi \in F$. By the inductive hypothesis we may assume that $e^F(v',\varphi) = v'(\varphi)$ for every $v' \in W$; thus we must prove

$$\inf_{v' \in W}\{S^F vv' \Rightarrow v'(\varphi)\} = v(\Box\varphi)) \tag{3.1}$$

$$\sup_{v' \in W}\{S^F vv' \cdot v'(\varphi)\} = v(\Diamond\varphi)) \tag{3.2}$$

By definition, $S^F vv' \leq (v(\Box\varphi) \Rightarrow v'(\varphi))$ and $S^F vv' \leq (v'(\varphi) \Rightarrow v(\Diamond\varphi))$ for any $\varphi \in F$ and $v' \in W$; therefore, $v(\Box\varphi) \leq (S^F vv' \Rightarrow v'(\varphi))$ and $S^F vv' \cdot v'(\varphi) \leq v(\Diamond\varphi)$. Taking the meet over v' in the first inequality and the join in the second,

$$v(\Box\varphi) \leq \inf_{v' \in W}\{S^F vv' \Rightarrow v'(\varphi)\}, \quad \sup_{v' \in W}\{S^F vv' \cdot v'(\varphi)\} \leq v(\Diamond\varphi).$$

Hence, if $v(\Box\varphi) = 1$ and $v(\Diamond\varphi) = 0$ we obtain immediately (3.1) and (3.2). Therefore, it only remains to prove the next two claims for $\Box\varphi, \Diamond\varphi \in F$.

Claim 1
If $v(\Box\varphi) = \alpha < 1$ and $\varepsilon > 0$, there exists a valuation $w \in W$ such that $S^F vw > w(\varphi)$ and $w(\varphi) < \alpha + \varepsilon$ (thus, $(S^F vw \Rightarrow w(\varphi)) < \alpha + \varepsilon$).

Claim 2
If $v(\Diamond\varphi) = \alpha > 0$ then, for any $\varepsilon > 0$, there exists $w \in W$ such that $w(\varphi) = 1$ and $S^F vw \geq \alpha - \varepsilon$ (thus $w(\varphi) \cdot S^F vw \geq \alpha - \varepsilon$).

PROOF OF CLAIM 1. By definition of $\Rightarrow$ in [0,1], to grant the required conditions on w it is necessary to find $w \in W$ and p_0 such that $\alpha + \varepsilon \geq p_0 > w(\varphi)$ and for any $\theta \in F$: $v(\Box\theta) \leq w(\theta)$ if $w(\theta) < p_0$, $w(\theta) \leq v(\Diamond\theta)$ if $v(\Diamond\theta) < p_0$. This is achieved in two stages: first producing a valuation $u \in W$ satisfying $u(\varphi) < 1$ and the relative ordering conditions the $w(\theta)$ must satisfy, conditions which may be coded by a theory $\Gamma_{\varphi,v}$, and then moving the values $u(\theta)$, $\theta \in F$, to the correct valuation w by composing u with an increasing bijection of [0,1]. Assume $v(\Box\varphi) = \alpha < 1$ and define (all formulas involved ranging in $\mathcal{L}_{\Box\Diamond}(Var)$)

$$\begin{aligned}\Gamma_{\varphi,v} = \; & \{\theta : v(\Box\theta) > \alpha\} \cup \{\theta_1 \to \theta_2 : v(\Diamond\theta_1) \leq v(\Box\theta_2)\} \\ & \cup \{(\theta_2 \to \theta_1) \to \theta_1 : v(\Diamond\theta_1) < v(\Box\theta_2)\}.\end{aligned}$$

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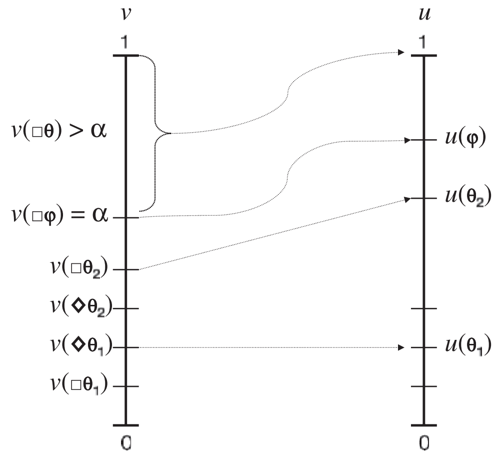

Obs: if $v(\Diamond\theta_1) < \alpha$ then $u(\theta_1) < u(\varphi)$

FIGURE 1. First translation.

Then we have $v(\Box\xi) > \alpha$ for each $\xi \in \Gamma_{\varphi,v}$: for the first set of formulas by construction, for the second because $v(\Box(\theta_1 \to \theta_2)) \geq v(\Diamond\theta_1 \to \Box\theta_2) = 1$ by FS2, and for the third, because $v(\Box\theta_2 \to \Diamond\theta_1) < 1$ and thus $v(\Box((\theta_2 \to \theta_1) \to \theta_1)) = 1$ by $T4$. This implies that

$$\Gamma_{\varphi,v}, \Sigma \nvdash_{\mathcal{G}_{\Box\Diamond}} \varphi.$$

Otherwise, $\xi_1,\ldots,\xi_k \in \Gamma_{\varphi,v}$ would exist such that $\xi_1,\ldots,\xi_k, \Sigma \vdash_{\mathcal{G}_{\Box\Diamond}} \varphi$. Hence, $\Box\xi_1,\ldots,\Box\xi_k$, $\Box\Sigma \vdash_{\mathcal{G}_{\Box\Diamond}} \Box\varphi$ by NR$_\Box$ and K$_\Box$,but $\Sigma \vdash_{\mathcal{G}_{\Box\Diamond}} \Box\Sigma$ by normality. Then $\Box\xi_1,\ldots,\Box\xi_k, \Sigma, Th\mathcal{G}_{\Box\Diamond} \vdash_{\mathcal{G}} \Box\varphi$ by Lemma 2.2 and thus by Proposition 2.1 (i), and recalling that $v(\Sigma \cup Th\mathcal{G}_{\Box\Diamond}) = 1$,

$$\alpha < \inf v(\{\Box\xi_1,\ldots,\Box\xi_k\} \cup \Sigma \cup Th\mathcal{G}_{\Box\Diamond}) \leq v(\Box\varphi) = \alpha,$$

a contradiction. Therefore, by Proposition 2.1 (ii) there exists a valuation $u : Var \cup X \mapsto [0,1]$ such that $u(\Gamma_{\varphi,v} \cup \Sigma \cup Th\mathcal{G}_{\Box\Diamond}) = 1$ and $u(\varphi) < 1$. This implies the following relations between v and u, that we list for further use (see Figure 1). Given $\theta_1, \theta_2, \theta_3$,

#**1**. If $v(\Box\theta) > \alpha$ then $u(\theta) = 1$ (since then $\theta \in \Gamma_{\varphi,v}$)
#**2** If $v(\Diamond\theta_1) \leq v(\Box\theta_2)$ then $u(\theta_1) \leq u(\theta_2)$ (since then $\theta_1 \to \theta_2 \in \Gamma_{\varphi,v}$)
#**3** If $v(\Diamond\theta_1) < v(\Box\theta_2)$ then $u(\theta_1) < u(\theta_2)$ or $u(\theta_1) = u(\theta_2) = 1$ (since then $(\theta_2 \to \theta_1) \to \theta_1) \in \Gamma_{\varphi,v}$)
#**4**. If $v(\Box\theta_2) > 0$ then $u(\theta_2) > 0$ (making $\theta_1 := \bot$ in #**3** since $u(\bot) = v(\Diamond\bot) = 0$).

For the next construction we need the finiteness of F. Set $B = \{v(\Box\theta) : \theta \in F\}$, for each $b \in B$ define

$$u_b = \min\{u(\theta) : \theta \in F \text{ and } v(\Box\theta) = b\},$$

and then define a strictly descending sequence $b_0, b_1, \ldots, b_N = 0$ in B as follows:

$b_0 = \alpha$
$b_{i+1} = \max\{b \in B : b < b_i \text{ and } u_b < u_{b_i}\}$.

Pick formulas $\varphi_i \in F$ such that $b_i = v(\Box\varphi_i)$ and $u_{b_i} = u(\varphi_i)$. By finiteness of B, the inductive definition ends with some b_N (which could be b_0 in case $u_\alpha = 0$). To check that $b_N = 0$, assume $b_N = v(\Box\varphi_N) > 0$,

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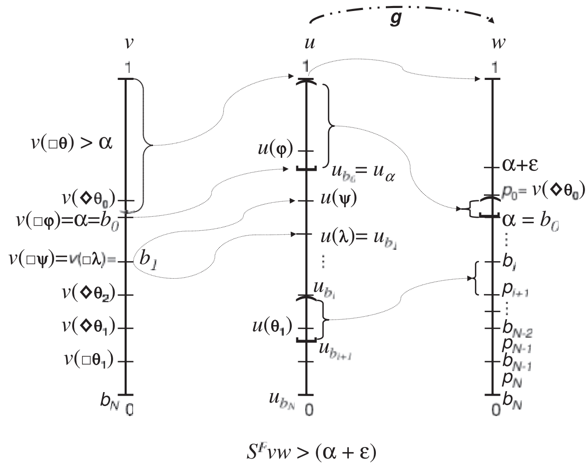

FIGURE 2. Second translation.

then $u_{b_N} = u(\varphi_N) > 0$ by property **#4** above. But $v(\Box\bot) \leq v(\Box\varphi_N)$ by $\mathrm{RN}_\Box$ and $\mathrm{K}_\Box$ and $u(\bot) = 0$. Thus, by minimality of u_{b_N} we cannot have equality; hence, $v(\Box\bot) < v(\Box\varphi_N)$ and thus there exists $b_{N+1} < b_N$, a contradiction.

By construction, the sequence $u_{b_0}, u_{b_1}, \ldots$ is also strictly descending with $u_{b_0} = u_\alpha \leq u(\varphi) < 1$, and it ends at 0 because $v(\Box\bot) \leq v(\Box\varphi_N) = 0$ and thus $u_{b_N} \leq u(\bot) = 0$ by minimality again.

Fix $\varepsilon > 0$ such that $\alpha + \varepsilon < 1$ and further define (taking $\min\emptyset = 1$)

$$p_0 = (\alpha+\varepsilon)\cdot \min\{v(\Diamond\theta) : \theta \in F,\ \alpha < v(\Diamond\theta)\}$$
$$p_{i+1} = b_i \cdot \min\{v(\Diamond\theta) : \theta \in F,\ b_{i+1} < v(\Diamond\theta)\} \text{ for } i \geq 1.$$

Notice that we have $p_i > b_i$ by construction.

Summing up,

$$1 > \alpha+\varepsilon \geq p_0 > b_0 = \alpha \geq p_1 > b_1 \geq \cdots \geq p_N > b_N = 0$$
$$1 > u_{b_0} > u_{b_1} > \cdots > u_{b_N} = 0.$$

Now pick a strictly increasing function $g : [0,1] \mapsto [0,1]$ such that (see Figure 2)

$$g(1) = 1$$
$$g[[u_\alpha, 1)] = [\alpha, p_0)$$
$$g[[u_{b_{i+1}}, u_{b_i})] = [b_{i+1}, p_{i+1}).$$

Then the valuation $w = g \circ u$ satisfies $w(\Sigma \cup Th\mathcal{G}_{\Box\Diamond}) = 1$, and so it belongs to W. Moreover, $w(\varphi) = g(u(\varphi)) < p_0 \leq \alpha + \varepsilon$. It remains to show that $S^F vw > w(\varphi)$. For any $\theta \in F$:

(i) If $u(\theta) = 1$ then $w(\theta) = 1$ by definition of w; hence, $v(\Box\theta) \leq w(\theta)$. In addition, $v(\Diamond\theta) \geq p_0$, otherwise $v(\Diamond\theta) \leq \alpha = v(\Box\varphi_0)$, which would imply $u(\theta) \leq u(\varphi_0) < 1$ by **#2**, a contradiction.

(ii) If $u(\theta) \in [u_{b_i}, u_{b_{i-1}})$ or $u(\theta) = [u_{b_0}, 1)$ then $v(\Box\theta) \leq w(\theta) \leq v(\Diamond\theta)$. To see this notice first that $w(\theta) \in [b_i, p_i)$ by definition of g. Now, for $i \geq 1$, b_i is the maximum $v(\Box\psi)$ with $u(\psi) < u_{b_{i-1}}$. Therefore, $v(\Box\theta) \leq b_i \leq w(\theta)$. In addition, for $i = 0$, $v(\Box\theta) \leq \alpha = b_0 \leq w(\theta)$ by **#1.** Moreover, if $u(\theta) = u_{b_i} = u(\varphi_i)$ then $w(\theta) = b_i = v(\Box\varphi_i) \leq v(\Diamond\theta)$ by the counter-reciprocal of **#3** because $u_{b_i} < 1$, and if $u(\theta) > u_{b_i}$ then $v(\Diamond\theta) > v(\Box\varphi_i) = b_i$ by the counter-reciprocal of **#2**; hence, $v(\Diamond\theta) \geq p_i > w(\theta)$.

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It follows form from (i,ii) that $\inf_{\theta\in F}\{v(\Box\theta)\Rightarrow w(\theta)\}=1$ and $\inf_{\theta\in F}\{w(\theta)\Rightarrow v(\Diamond\theta)\}\geq p_0$. Hence, $S^F vw\geq p_0>w(\varphi)$. This finishes the proof of Claim 1.

PROOF OF CLAIM 2. Again we code first in a relative consistence situation the minimal requirements for w, to obtain $u\in W$ satisfying those requirements, and then transform u to the correct w by an automorphism of [0,1]. Assume $v(\Diamond\varphi)=\alpha>0$ and define

$$\begin{aligned}U_{\varphi,v}=\ &\{\theta: v(\Diamond\theta)<\alpha\}\\ &\cup\{\vartheta_2\to\vartheta_1: v(\Diamond\vartheta_1)<v(\Box\vartheta_2)\text{ and }v(\Diamond\vartheta_1)<\alpha\}\\ &\cup\{(\vartheta_1\to\vartheta_2)\to\vartheta_1: v(\Diamond\vartheta_1)=v(\Box\vartheta_2)\text{ and }v(\Diamond\vartheta_1)<\alpha\}.\end{aligned}$$

This set is non-empty because $v(\Diamond\bot)=0$. Moreover, for any $\xi\in U_{\varphi,v}$ we have $v(\Diamond\xi)<\alpha$; for the first set of axioms by construction; for the second because $v(\Diamond(\vartheta_2\to\vartheta_1))\leq v(\Box\vartheta_2\to\Diamond\vartheta_1)=v(\Diamond\vartheta_1)<\alpha$ by $FS1$; and for the third because $v(\Diamond((\vartheta_1\to\vartheta_2)\to\vartheta_1)))\leq v(\Box(\vartheta_1\to\vartheta_2)\to\Diamond\vartheta_1)\leq v((\Diamond\vartheta_1\to\Box\vartheta_2)\to\Diamond\vartheta_1)=v(\Diamond\vartheta_1)<\alpha$ by FS1, FS2.

We claim that for any finite $\{\xi_1,\ldots,\xi_k\}\subseteq U_{\varphi,v}$:

$$\varphi,\Sigma\nvdash_{G_{\Box\Diamond}}\xi_1\vee\ldots\vee\xi_k$$

because, on the contrary, $\Sigma\vdash_{G_{\Box\Diamond}}\Diamond\varphi\to\Diamond(\xi_1\vee\ldots\vee\xi_k)\vdash_{G_{\Box\Diamond}}\Diamond\varphi\to(\Diamond\xi_1\vee\ldots\vee\Diamond\xi_k)$ by normality of Σ and $K_\Diamond$, whence

$$\Diamond\varphi,\Sigma,Th\mathcal{G}_{\Box\Diamond}\vdash\Diamond\xi_1\vee\ldots\vee\Diamond\xi_k,$$

and evaluating with v it would give: $\alpha=\inf v(\{\Diamond\varphi\}\cup\Sigma\cup Th\mathcal{G}_{\Box\Diamond})\leq\max\{v(\Diamond\xi_1),\ldots,v(\Diamond\xi_k)\}<\alpha$, absurd.

Therefore, there is a valuation u such that $u(\varphi)=u(\Sigma\cup T\mathcal{G}_{\Box\Diamond})=1$ and $u(\xi)<1$ for each $\xi\in U_{\varphi,v}$, which has the following consequences for any θ,θ_1,θ_2:

##**1**. If $v(\Diamond\theta)<\alpha$ then $u(\theta)<1$ (because then $\theta\in U_{\varphi,.v}$)
##**2**. If $v(\Diamond\theta_1)<v(\Box\theta_2)$ and $v(\Diamond\theta_1)<\alpha$ then $u(\theta_1)<u(\theta_2)$ (because $\theta_2\to\theta_1\in U_{\varphi,.v}$)
##**3**. If $v(\Diamond\theta_1)\leq v(\Box\theta_2)$ and $v(\Diamond\theta_1)<\alpha$ then $u(\theta_1)\leq u(\theta_2)$ (because $(\theta_1\to\theta_2)\to\theta_1\in U_{\varphi,v}$)
##**4** If $u(\theta_2)=0$ then $v(\Box\theta_2)=0$ (making $\theta_1:=\bot$ in ##**2** and taking counter-reciprocal)
##**5**. If $v(\Diamond\theta_1)=0$ then $u(\theta)=0$ (making $\theta_2:=\bot$ in ##**3**, because then $v(\Diamond\theta_1)\leq v(\Box\bot)$ and $v(\Diamond\theta_1)<\alpha$).

We perform now a dual construction of the one we made in the proof of Claim 1. Let $C=\{v(\Diamond\theta)\leq\alpha:\theta\in F\}$ and define for each $c\in C$

$$u_c=\max\{u(\theta):\theta\in F,\ v(\Diamond\theta)=c\}.$$

Note that $u_0=0$ by **##5** above, and $u_\alpha=1$ because $u(\varphi)=1$. Define an ascending sequence $0=c_0<c_1<\ldots.$ in C as follows:

$c_0=v(\Diamond\bot)=0$
$c_1=\min\{c\in C: c>c_0\text{ and }u_c>u_{c_0}\}$
$c_2=\min\{c\in C: c>c_1\text{ and }u_c>u_{c_1}\}$
etc.

Choose φ_i such that $u_{c_i}=u(\varphi_i)$, $c_i=v(\Diamond\varphi_i)$. Clearly, $0=u_{c_0}<u_{c_1}<\ldots.$ By finiteness of F the sequence of the c_i ends necessarily with $c_N=\alpha$, because $c_i=v(\Diamond\varphi_i)<\alpha$ implies $u_{c_i}=u(\varphi_i)<1=u_\alpha$ by ##**1** above and thus the existence of $c_{i+1}\leq\alpha$. This means also that $u_{c_n}=1$.

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Fix $\varepsilon > 0$ such that $\alpha - \varepsilon > c_{N-1}$, and further define (taking $\max\emptyset = 0$)

$$q_{N-1} = \max\{\alpha - \varepsilon, \max\{v(\Box\theta) : v(\Box\theta) < c_N\}\}$$
$$q_i = \max\{c_i, \max\{v(\Box\theta) : v(\Box\theta) < c_{i+1}\}\}, \text{ for } i < N-1.$$

Then we have:

$$0 = c_0 \leq q_0 < c_1 \leq q_1 < \ldots . c_{N-1} \leq \alpha - \varepsilon \leq q_{N-1} < c_N = \alpha$$
$$0 = u_{c_0} < u_{c_1} < \ldots .. < u_{c_N} = 1.$$

Choose $g : [0,1] \to [0,1]$ to be any strictly increasing function such that

$$g(0) = 0$$
$$g[(u_{c_i}, u_{c_{i+1}}]] = (q_i, c_{i+1}] \text{ for } i < N-1$$
$$g[(u_{c_{N-1}}, 1)] = (q_{N-1}, \alpha)$$
$$g(1) = 1$$

Then g is a Heyting homomorphism and the valuation $w = g \circ v$ satisfies $w(\varphi) = w(\Sigma \cup T\mathcal{G}_{\Box\Diamond}) = 1$; thus $w \in W$. It remains to show that $S^F vw \geq \alpha - \varepsilon$. Indeed, we have:

(i) If $v(\Diamond\theta) \geq \alpha$ then trivially $(w(\theta) \Rightarrow v(\Diamond\theta)) \geq \alpha$. In particular, $(w(\varphi) \Rightarrow v(\Diamond\varphi)) = (1 \Rightarrow v(\Diamond\varphi)) = \alpha$.

(ii) If $v(\Diamond\theta) < \alpha$ then $w(\theta) \leq v(\Diamond\theta)$. To see this consider cases. First: $u(\theta) \in (u_{c_i}, u_{c_{i+1}})$ for some i (recall $u(\theta) < 1$ by **##1**), then $w(\theta) \in (q_i, c_{i+1}]$. As $u(\theta) > u_{c_i}$ and $c_{i+1} = v(\Diamond\varphi_{i+1})$ is the smallest $v(\Diamond\psi)$ with $u(\psi) > u_{c_i}$ then $v(\Diamond\theta) \geq c_{i+1} \geq w(\theta)$. Second: $u(\theta) = 0$, then $w(\theta) = 0$ and $v(\Box\theta) = 0$ by **##4**.

(iii) If $v(\Box\theta) \geq \alpha$ then $(v(\Box\theta) \Rightarrow w(\theta)) > \alpha - \varepsilon$, because $v(\Box\theta) > c_{N-1} = v(\Diamond\varphi_{N-1})$ which implies $u(\theta) > u(\varphi_{N-1}) = u_{c_{N-1}}$ by **##2.** Therefore, $w(\theta) > q_{N-1} \geq \alpha - \varepsilon$ by definition.

(iv) $v(\Box\theta) < \alpha$ then $v(\Box\theta) \leq w(\theta)$. To see this notice that $c_i \leq v(\Box\theta) \leq q_i < c_{i+1}$ for some i and consider cases. First: $v(\Box\theta) = c_i = v(\Diamond\varphi_i)$ then, by **##3,** $u_{c_i} = u(\varphi_i) \leq u(\theta)$. Therefore $c_i \leq w(\theta)$. That is, $v(\Box\theta) \leq w(\theta)$. Second: $c_i < v(\Box\theta)$ then $u_{c_i} < u(\theta)$, by **##2,** and by definition $q_i \leq w(\theta)$, which shows again $v(\Box\theta) \leq w(\theta)$.

From (i,ii) we have: $\inf_{\theta \in F}\{w(\theta) \Rightarrow v(\Diamond\theta)\} = \alpha$, and from (iii,iv): $\inf_{\theta \in F}\{v(\Box\theta) \Rightarrow w(\theta)\} \geq \alpha - \varepsilon$. Hence, $S^F vw \geq \alpha - \varepsilon$. ■

Lemma 3.2 (Weak completeness)
For any finite theory T and formula φ in $\mathcal{L}_{\Box\Diamond}$, $T \models_{GK} \varphi$ implies $T \vdash_{\mathcal{G}_{\Box\Diamond}} \varphi$.

Proof. Assume T is finite and $T \nvdash_{\mathcal{G}_{\Box\Diamond}} \varphi$. Then $T, Th\mathcal{G}_{\Box\Diamond} \nvdash_{\mathcal{G}} \varphi$ by Lemma 2.2 and thus there is, by Proposition 2.1, a Gödel valuation $v : Var \cup X \to [0,1]$ such that $v(\varphi) < v(T) = v(Th\mathcal{G}_{\Box\Diamond}) = 1$. Let F be a finite fragment containing $T \cup \{\varphi\}$, then v is a world of the canonical model $M_{\emptyset,F}$ and by Lemma 3.1, $e^F(v,T) = v(T) = 1$ and $e^F(v,\varphi) = v(\varphi) < 1$. Thus $T \not\models_{GK} \varphi$. ■

To prove strong completeness we utilize compactness of first order classical logic and the following result of Horn:

Lemma 3.3 ([20], Lemma 3.7)
Any countable linear order $(P,<)$ may be embedded in $(\mathbb{Q} \cap [0,1], <)$ preserving all joins and meets existing in P.

Theorem 3.1 (Strong completeness)
For any countable theory T and formula φ in $\mathcal{L}_{\Box\Diamond}$, $T \vdash_{\mathcal{G}_{\Box\Diamond}} \varphi$ if and only if $T \models_{GK} \varphi$.

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PROOF. One direction of the equivalence follows from Theorem 2.1 (soundness). For the other direction, assume $T \nvdash_{\mathcal{G}_{\Box\Diamond}} \varphi$ and consider the first order theory T^* with two unary relation symbols W, P, a binary relation symbol $<$, three constant symbols $0, 1, c$, two binary function symbols $\circ$, S and a unary function symbol f_θ for each $\theta \in \mathcal{L}_{\Box\Diamond}(V)$, where V is the set of propositional variables occurring in formulas of T, and having for axioms:

$\forall x \neg (W(x) \wedge P(x))$
$\forall x (W(x) \vee \neg W(x))$
'$(P, <)$ is a strict linear order with minimum 0 and maximum 1'
$\forall x \forall y (W(x) \wedge W(y) \rightarrow P(S(x,y)))$
$\forall x \forall y (P(x) \wedge P(y) \rightarrow (x \leq y \wedge x \circ y = 1) \vee (x > y \wedge x \circ y = y))$
$\forall x (W(x) \rightarrow f_\perp(x) = 0)$
for each $\theta, \psi \in \mathcal{L}_{\Box\Diamond}$ the sentences:
$\forall x (W(x) \rightarrow P(f_\theta(x)))$
$\forall x (W(x) \rightarrow f_{\theta \wedge \psi}(x) = \min\{f_\theta(x), f_\psi(x)\})$
$\forall x (W(x) \rightarrow f_{\theta \rightarrow \psi}(x) = (f_\varphi(x) \circ f_\psi(x))$
$\forall x (W(x) \rightarrow f_{\Box\theta}(x) = \inf_y (S(x,y) \circ f_\theta(y))$
$\forall x (W(x) \rightarrow f_{\Diamond\theta}(x) = \sup_y (\min\{S(x,y), f_\theta(y)\})$
for each $\delta \in T$ the sentence: $f_\delta(c) = 1$
finally, $W(c) \wedge (f_\varphi(c) < 1)$.

For each finite part t of T^* let F be a finite fragment of $\mathcal{L}_{\Box\Diamond}$ containing $\{\theta : f_\theta$ occurs in $t\}$. Since $F \cap T \nvdash_{\mathcal{G}_{\Box\Diamond}} \varphi$ by hypothesis, then, by weak completeness, there is a GK-model $M_{\emptyset,F} = (W, S^F, e^F)$ and $a \in W$ such that $e^F(a,\theta) = 1$ for each $\theta \in F \cap T$ and $e^F(a,\varphi) < 1$. Therefore, the first order structure $(W \sqcup [0,1], W, [0,1], <, 0, 1, a, \Rightarrow, S^F, f_\theta)_{\theta \in \mathcal{L}_{\Box\Diamond}}$, with $f_\theta : W \rightarrow [0,1]$ defined as $f_\theta(x) = e^F(x,\theta)$, is clearly a model of t. By compactness of first order logic and the downward Löwenheim theorem, T^* has a countable model $M^* = (B, W, P, <, 0, 1, a, \circ, S, f_\theta)_{\theta \in \mathcal{L}_{\Box\Diamond}}$. Using Horn's lemma, $(P, <)$ may be embedded in $(\mathbb{Q} \cap [0,1], <)$ preserving $0, 1$, and all suprema and infima existing in P; therefore, we may assume without loss of generality that the ranges of the functions S and f_θ are contained in $[0,1]$. Then, it is straightforward to verify that $M = (W, S, e)$, where $e(w,\theta) = f_\theta(w)$ for all $w \in W$ and $\theta \in \mathcal{L}_{\Box\Diamond}(V)$, is a GK-model with a distinguished world a such that $M \models_a T$, and $M \not\models_a \varphi$. Hence, $T \not\models_{GK} \varphi$. ■

For normal theories we obtain a finer result:

THEOREM 3.2
If T is a countable normal theory there is GK-model M_T such that for any φ: $T \vdash_{\mathcal{G}_{\Box\Diamond}} \varphi$ if and only if $M_T \models \varphi$.

PROOF. Assume $T \nvdash_{\mathcal{G}_{\Box\Diamond}} \varphi$. Then, by Lemma 3.1, for each finite fragment F of $\mathcal{L}_{\Box\Diamond}(V)$ containing φ the canonical model $M_{T,F}$ is such that $M_{T,F} \models T \cap F$ and $M_{T,F} \not\models \varphi$. Add to the theory T^* in the proof of Theorem 3.1 the sentence $\forall x (W(x) \rightarrow f_\delta(x) = 1)$ for each $\delta \in T$. Then by the previous observation each finite part of T^* has a model. Arguing as in the quoted proof, we obtain a GK-model $M_\varphi = \langle W_\varphi, S_\varphi, e_\varphi \rangle$ such that $e_\varphi(w, T) = 1$ for all $w \in W_\varphi$ and $e(w_\varphi, \varphi) < 1$. Define now $M_T = (W, S, e)$ where $W = \amalg_\varphi \{W_\varphi : T \nvdash_{\mathcal{G}_{\Box\Diamond}} \varphi\}$, $Sww' = S_\varphi ww'$ if $w, w' \in W_\varphi$ and 0 otherwise, and $e(w,p) = e_\varphi(w,p)$ for $w \in W_\varphi$. It is easily verified by induction on the complexity of θ that $e(w,\theta) = e_\varphi(w,\theta)$ for any $w \in W_\varphi$. Thus, $M_T \models T$ and hence $T \vdash_{\mathcal{G}_{\Box\Diamond}} \varphi$ implies $M_T \models \varphi$ by soundness; reciprocally, if $T \nvdash_{\mathcal{G}_{\Box\Diamond}} \varphi$ then $e(w_\varphi, \varphi) = e_\varphi(w_\varphi, \varphi) < 1$ by construction, and thus $M_T \not\models \varphi$. ■

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We cannot expect similar results for uncountable theories by the observation after Proposition 2.1. In fact,

PROPOSITION 3.1
There is no single linearly ordered Heyting algebra H giving strong completeness with respect to HK-models for theories of arbitrary power, even in Gödel–Dummett logic.

PROOF. Assume otherwise, then H would be infinite (by the known Gödel argument). Let κ be a cardinal greater than $|H|$ and consider the theory $T=\{(p_\beta \to p_\alpha)\to q : \alpha<\beta<\kappa\}$. Then $T\models_{HK} q$, because $v(T)=1$ with $v(q)<1$ would imply $v(p_\beta \to p_\alpha)<1$, and thus $v(p_\alpha)<v(p_\beta)$ for all $\alpha<\beta<\kappa$, yielding a subset of H of power κ, which is impossible by hypothesis. On the other hand, $T\nvdash_{\mathcal{G}_{\Box\Diamond}} q$. Otherwise, we would have $\Delta\vdash_{\mathcal{G}_{\Box\Diamond}} q$ and thus $\Delta\models_{HK} q$, for some finite set $\Delta=\{(p_{\alpha_{i+1}}\to p_{\alpha_i})\to q : 1\leq i<n\}$, which is impossible because the valuation $v(p_{\alpha_i})=h_i$, $v(q)=h$, where $h_1<h_2<\ldots<h_{n+1}<h<1$ makes $v((p_{\alpha_{i+1}}\to p_{\alpha_i})\to q)=1$ for $1\leq i<n$. ■

4 Modal axioms, optimal models

The notions and results in this section make sense and hold for HK-models where H is any complete Heyting algebra. Thus we state and prove them in this general framework.

Call a HK-frame $\mathcal{M}=\langle W,S\rangle$ *reflexive* if $Sxx=1$ for all $x\in W$, *transitive* if $Sxy\cdot Syz\leq Sxz$ for all $x,y,z\in W$, *symmetric* if $Sxy=Syx$ for all $x,y\in W$, and *euclidean* if $Sxy\cdot Sxz\leq Syz$ for all $x,y,z\in W$.

Let *Ref*, *Trans*, *Symm* and *Euclid* denote, respectively, the classes of HK- models over frames satisfying, respectively, each one of the above properties. These are the fuzzy versions of the corresponding properties of classical frames, classically characterized by the following pairs of modal schemes:

$$\begin{array}{llll} \mathrm{T}_\Box\ \ \Box\varphi\to\varphi & \mathrm{T}_\Diamond\ \ \varphi\to\Diamond\varphi & \text{reflexivity} \\ 4_\Box\ \ \Box\varphi\to\Box\Box\varphi & 4_\Diamond\ \ \Diamond\Diamond\varphi\to\Diamond\varphi & \text{transitivity} \\ \mathrm{B}_1\ \ \varphi\to\Box\Diamond\varphi & \mathrm{B}_2\ \ \Diamond\Box\varphi\to\varphi & \text{symmetry} \\ \mathrm{E}_1\ \ \Diamond\varphi\to\Box\Diamond\varphi & \mathrm{E}_2\ \ \Diamond\Box\varphi\to\Box\varphi & \text{euclidean property} \end{array} \tag{4.1}$$

LEMMA 4.1
(i) $T_\Box$ and $T_\Diamond$ are valid in *Ref*. (ii) $4_\Box$ and $4_\Diamond$ are valid in *Trans*. (iii) B_1 and B_2 are valid in *Symm*. (iv) E_1 and E_2 are valid in *Euclid*.

PROOF. (i) In reflexive models, $e(x,\Box\varphi)\leq(Sxx\Rightarrow e(x,\varphi))=e(x,\varphi)=Sxx\cdot e(x,\varphi)\leq e(x,\Diamond\varphi)$ for any x. Thus $e(x,\Box\varphi\to\varphi)=1=e(x,\varphi\to\Diamond\varphi)$.

(ii) In transitive models, $e(x,\Box\varphi)\cdot Sxy\cdot Syz\leq[(Sxz\Rightarrow e(z,\varphi))\cdot Sxz]\leq e(z,\varphi)$ for all x,y,z. Hence, $e(x,\Box\varphi)\cdot Sxy\leq(Syz\Rightarrow e(z,\varphi))$ and thus $e(x,\Box\varphi)\cdot Sxy\leq e(y,\Box\varphi)$. Therefore, $e(x,\Box\varphi)\leq(Sxy\Rightarrow e(y,\Box\varphi))$ for all y and thus $e(x,\Box\varphi)\leq e(x,\Box\Box\varphi)$ which yields $4_\Box$. Also $Sxy\cdot Syz\cdot e(z,\varphi)\leq Sxz\cdot e(z,\varphi)\leq e(x,\Diamond\varphi)$. Hence, $Syz\cdot e(z,\varphi)\leq(Sxy\Rightarrow e(x,\Diamond\varphi))$ and thus $e(x,\Diamond\varphi)\leq(Sxy\Rightarrow e(x,\Diamond\varphi))$. Therefore, $Sxy\cdot e(x,\Diamond\varphi)\leq e(x,\Diamond\varphi))$ for all y and thus $e(x,\Diamond\Diamond\varphi)\leq e(x,\Diamond\varphi)$ which gives $4_\Diamond$.

(iii) In symmetric models, $Sxy\cdot e(x,\varphi)=Syx\cdot e(x,\varphi)\leq e(y,\Diamond\varphi)$ for all x,y. Then $e(x,\varphi)\leq(Sxy\Rightarrow e(y,\Diamond\varphi))$ and thus $e(x,\varphi)\leq e(x,\Box\Diamond\varphi))$ which is B_1. Moreover, $e(y,\Box\varphi)\leq(Syx\Rightarrow e(x,\varphi))$; then $Sxy\cdot e(y,\Box\varphi)=Syx\cdot e(y,\Box\varphi)\leq e(x,\varphi)$ and thus B_2 follows.

(iv) In euclidean models, $Sxy\cdot e(y,\varphi)\cdot Sxz=Szy\cdot e(y,\varphi)\leq e(z,\Diamond\varphi)$ for all x,y,z. Then $Sxy\cdot e(y,\varphi)\leq(Sxz\Rightarrow e(z,\Diamond\varphi))$ and E_1 follows. Similarly, $Sxy\cdot e(y,\Box\varphi)\cdot Sxz\leq Syz\cdot(Syz\Rightarrow e(z,\varphi))\leq e(z,\varphi))$, and thus $Sxy\cdot e(y,\Box\varphi)\leq(Sxz\Rightarrow e(z,\varphi))$, from which E_2 follows. ■

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To extend the completeness theorem to the [0,1]-valued analogues of the classical bi-modal systems T, S4, S5, we introduce a particular kind of GK-model, their advantage being that the many-valued counterpart of classical structural properties of frames may be characterized in them by the validity of the corresponding classical schemes.

DEFINITION 4.1
Given a HK-model $M=(W,S,e)$, define a new accessibility relation $S^+xy = S_\Box xy\cdot S_\Diamond xy$, where $S_\Box xy=\inf_{\varphi\in\mathcal{L}_{\Box\Diamond}}\{e(x,\Box\varphi)\Rightarrow e(y,\varphi)\}$ and $S_\Diamond xy=\inf_{\varphi\in\mathcal{L}_{\Box\Diamond}}\{e(y,\varphi)\Rightarrow e(x,\Diamond\varphi)\}$. Call M *optimal* if $S^+=S$.

The following lemma shows that any model is equivalent to an optimal one.

LEMMA 4.2
(W,S^+,e) is optimal. If e^+ is the extension of e in this model then $e^+(x,\varphi)=e(x,\varphi)$ for any $\varphi\in\mathcal{L}_{\Box\Diamond}$.

PROOF. The first claim follows from the second (which implies $S^{++}=S^+$), and the second is proven by induction on the complexity of formulas. The only non-trivial step is that of the modal connectives. Notice first that $Sxy\le S^+xy$, because $e(x,\Box\varphi)\le(Sxy\Rightarrow e(y,\varphi))$ and $Sxy\cdot e(y,\varphi)\le e(x,\Diamond\varphi)$ for any φ; thus $Sxy\le(e(x,\Box\varphi)\Rightarrow e(y,\varphi))$, $(e(y,\varphi)\Rightarrow e(x,\Diamond\varphi))$. Now, assume $e^+(y,\varphi)=e(y,\varphi)$ for all y, then by the previous observation and the induction hypothesis: $e^+(x,\Box\varphi)=\inf_y\{S^+xy\Rightarrow e^+(y,\varphi)\}\le\inf_y\{Sxy\Rightarrow e(y,\varphi)\}=e(x,\Box\varphi)$. But $S^+xy\le(e(x,\Box\varphi)\Rightarrow e(y,\varphi))$ by definition of S^+ and thus $e(x,\Box\varphi)\le(S^+xy\Rightarrow e(y,\varphi))=(S^+xy\Rightarrow e^+(y,\varphi))$ which yields $e(x,\Box\varphi)\le e^+(x,\Box\varphi)$. Similarly, by the induction hypothesis and the first observation, $e^+(x,\Diamond\varphi)=\sup_y\{S^+xy\cdot e(y,\varphi)\}\ge\sup_y\{Sxy\cdot e(y,\varphi)\}=e(x,\Diamond\varphi)$, and by definition $S^+xy\le(e(y,\varphi)\Rightarrow e(x,\Diamond\varphi))$. Thus $S^+xy\cdot e^+(y,\varphi)=S^+xy\cdot e(y,\varphi)\le e(x,\Diamond\varphi)$ which yields $e^+(x,\Diamond\varphi)\le e(x,\Diamond\varphi)$. ■

PROPOSITION 4.1
An optimal HK-model is: (i) reflexive if and only if it validates the schemes $T_\Box+T_\Diamond$, (ii) transitive if and only if it validates $4_\Box+4_\Diamond$, (iii) symmetric if and only if it validates B_1+B_2, (iv) euclidean if and only if it validates E_1+E_2.

PROOF. (i) If $T_\Box$ and $T_\Diamond$ hold, $Sxx=\inf_\varphi\{e(x,\Box\varphi\to\varphi)\}\cdot\inf_\varphi\{e(x,\varphi\to\Diamond\varphi)\}=1$, by optimality.
(ii) $S_\Box xy\cdot S_\Box yz\le(e(x,\Box\Box\varphi)\Rightarrow e(y,\Box\varphi))\cdot(e(y,\Box\varphi)\Rightarrow e(z,\varphi))\le(e(x,\Box\Box\varphi)\Rightarrow e(z,\varphi))\le(e(x,\Box\varphi)\Rightarrow e(z,\varphi))$, the last inequality holding by $4_\Box$. Similarly, $S_\Diamond xy\cdot S_\Diamond yz\le(e(y,\Diamond\varphi)\Rightarrow e(x,\Diamond\Diamond\varphi))\cdot(e(z,\varphi)\Rightarrow e(y,\Diamond\varphi))\le(e(z,\varphi)\Rightarrow e(x,\Diamond\Diamond\varphi))\le(e(z,\varphi)\Rightarrow e(x,\Diamond\varphi))$, the last inequality holding by $4_\Diamond$. Hence, $Sxy\cdot Syz\le(e(x,\Box\varphi)\Rightarrow e(z,\varphi))\cdot(e(z,\varphi)\Rightarrow e(x,\Diamond\varphi)$ by optimality. Taking meet over φ in the right we get transitivity.

(iii) Since $S_\Box xy\le(e(x,\Box\Diamond\varphi)\Rightarrow e(y,\Diamond\varphi))\le(e(x,\varphi)\Rightarrow e(y,\Diamond\varphi))$ by B_1, then taking meet over φ, we obtain $S_\Box xy\le S_\Diamond yx$. Similarly, $S_\Diamond yx\le(e(x,\Box\varphi)\Rightarrow e(y,\Diamond\Box\varphi))\le(e(x,\Box\varphi)\Rightarrow e(y,\varphi))$ by B_2, and then $S_\Diamond yx\le S_\Box xy$. From this, $S_\Diamond xy=S_\Box yx$, and thus $Sxy=Syx$.

(iv) Assuming E_1, $S_\Diamond xz\le(e(z,\varphi)\Rightarrow e(z,\Diamond\varphi))\le(e(z,\varphi)\Rightarrow e(z,\Box\Diamond\varphi))$ for any formula φ, and combining this with $S_\Box xy\le(e(x,\Box\Diamond\varphi)\Rightarrow e(y,\Diamond\varphi))$, we obtain $S_\Box xy\cdot S_\Diamond xz\le(e(z,\varphi)\Rightarrow e(z,\Diamond\varphi))$. Similarly, assuming E_2, $S_\Diamond xy\le(e(y,\Box\varphi)\Rightarrow e(x,\Diamond\Box\varphi))\le(e(y,\Box\varphi)\Rightarrow e(x,\Box\varphi))$, and combining this with $S_\Box xz\le(e(x,\Box\varphi)\Rightarrow e(z,\varphi))$, we obtain $S_\Diamond xy\cdot S_\Box xz\le(e(y,\Box\varphi)\Rightarrow e(z,\varphi))$. Multiplying the obtained inequalities we get: $Sxy\cdot Sxz\le(e(y,\Box\varphi)\Rightarrow e(z,\varphi))\cdot(e(z,\varphi)\Rightarrow e(z,\Diamond\varphi))$, which yields $Sxy\cdot Sxz\le Syz$ by optimality. ■

REMARK. Another relevant property of classical Kripke frames is *seriality:* $\forall x\exists y Sxy=1$, characterized (classically) by any of the axioms $\Diamond\top$ or $\neg\Box\bot$. We have not been able to characterize this

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property in GK-frames. However, its fuzzy version: $\forall x \sup_{y\in W} Sxy = 1$, is readily seen to be equivalent in arbitrary HK-frames to the validity of $\Diamond\top$, while the axiom $\neg\Box\bot$ characterizes only the weaker condition $\forall x \exists y Sxy > 0$.

5 Gödel analogues of classical bi-modal systems

Lemma 4.2, in conjunction with Proposition 4.1 and Theorem 3.1, implies strong completeness of any combination of axiom pairs in Table 4.1, with respect to GK-models over frames satisfying the associated structural properties. In particular, we obtain completeness for the Gödel analogues of the classical modal systems T, $S4$ and $S5$:

$$\begin{aligned}
\mathcal{G}\mathrm{T}_{\Box\Diamond} &:= \mathcal{G}_{\Box\Diamond} + \mathrm{T}_{\Box} + \mathrm{T}_{\Diamond}\\
\mathcal{G}\mathrm{S4}_{\Box\Diamond} &:= \mathcal{G}_{\Box\Diamond} + \mathrm{T}_{\Box} + \mathrm{T}_{\Diamond} + 4_{\Box} + 4_{\Diamond}\\
\mathcal{G}\mathrm{S5}_{\Box\Diamond} &:= \mathcal{G}_{\Box\Diamond} + \mathrm{T}_{\Box} + \mathrm{T}_{\Diamond} + 4_{\Box} + 4_{\Diamond} + \mathrm{B}_1 + \mathrm{B}_2
\end{aligned}$$

These systems may be seen to be equivalent, respectively, to the purely intutionistic modal logics IT, $IS4$, $IS5 = MIPC$ ([14], [28], [4]) plus the prelinearity scheme. We let the reader consider other relevant combinations. Recall that strong completeness refers here to entailment from countable theories.

THEOREM 5.1
$\mathcal{G}T_{\Box\Diamond}$ is strongly complete for $\models_{GK\cap Ref}$.
$\mathcal{G}S4_{\Box\Diamond}$ is strongly complete for $\models_{GK\cap Ref\cap Trans}$.
$\mathcal{G}S5_{\Box\Diamond}$ is strongly complete for $\models_{GK\cap Ref\cap Trans\cap Symm}$.

PROOF. If $T \models_{GK\cap Ref} \varphi$ then $T \models_{GK\cap \mathrm{Optimal}\cap Ref} \varphi$. Thus $T + \{\mathrm{T}_{\Box}, \mathrm{T}_{\Diamond}\} \models_{GK\cap\mathrm{Optimal}} \varphi$ by Proposition 4.1, and $T + \{\mathrm{T}_{\Box}, \mathrm{T}_{\Diamond}\} \models_{\mathrm{GK}} \varphi$ by Lemma 4.2. Therefore, $T + \{\mathrm{T}_{\Box}, \mathrm{T}_{\Diamond}\} \vdash_{\mathcal{G}_{\Box\Diamond}} \varphi$ by 3.1, which implies $T \vdash_{\mathcal{G}T_{\Box\Diamond}} \varphi$. The proofs of the other two cases are similar. ■

We focus on the system $\mathcal{G}\mathrm{S5}_{\Box\Diamond}$ that may be considerably simplified because the symmetry axioms B_1 and B_2 imply the inter-deducibility of each pair {FS1,FS2}, $\{\mathrm{T}_{\Box}, \mathrm{T}_{\Diamond}\}$ and $\{4_{\Box}, 4_{\Diamond}\}$, and $\{\mathrm{B}_2, \mathrm{P}\}$ deduces $\mathrm{F}_{\Diamond}$. Moreover, in the presence of $\{\mathrm{T}_{\Box}, \mathrm{T}_{\Diamond}\}$, the euclidean axioms $\{\mathrm{E}_1, \mathrm{E}_2\}$ are equivalent to $\{4_{\Box}, 4_{\Diamond}\} + \{\mathrm{B}_1, \mathrm{B}_2\}$; therefore, we are left with the modal axioms:

$$\begin{aligned}
&\Box(\varphi \to \psi) \to (\Box\varphi \to \Box\psi)\\
&\Diamond(\varphi \vee \psi) \to (\Diamond\varphi \vee \Diamond\psi)\\
&\Box(\varphi \to \psi) \to (\Diamond\varphi \to \Diamond\psi)\\
&\Box\varphi \to \varphi\\
&\varphi \to \Diamond\varphi\\
&\Diamond\varphi \to \Box\Diamond\varphi\\
&\Diamond\Box\varphi \to \Box\varphi
\end{aligned}$$

$\mathcal{G}\mathrm{S5}_{\Box\Diamond}$ presents some features that distinguish it from the weaker systems $\mathcal{G}_{\Box\Diamond}$, $\mathcal{G}\mathrm{T}_{\Box\Diamond}$ and $\mathcal{G}\mathrm{S4}_{\Box\Diamond}$. The uni-modal fragments of the latter logics have simple axiomatizations while axiomatizations for the uni-modal fragments of $\mathcal{G}\mathrm{S5}_{\Box\Diamond}$ are unknown. The $\Box$-fragments of the weaker systems are characterized by their accessibility-crisp models, as shown in [6], but this is not the case for the $\Box$-fragments of $\mathcal{G}\mathrm{S5}_{\Box\Diamond}$, as the following example illustrates.

EXAMPLE. The formula $\Box(\Box\varphi \vee \psi) \to (\Box\varphi \vee \Box\psi)$ is not a theorem of $\mathcal{G}\mathrm{S5}_{\Box}$ but it is valid in any accessibility-crisp model of $\mathcal{G}\mathrm{S5}_{\Box}$. The first claim is granted by the following two worlds model:

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$$\{p=\tfrac{1}{2},\ q=1\}\ \underset{\overset{\circlearrowleft}{1}}{u} \underset{\frac{1}{2}}{\longleftrightarrow} \underset{\overset{\circlearrowleft}{1}}{v}\ \{p=\tfrac{1}{2},\ q=0\}$$

in which the reader may verify that $e(u,\Box(\Box p\vee q))=1$ and $e(u,\Box p\vee\Box q)=\frac{1}{2}$. To verify the second claim notice that if $(W,S,e)\in Ref\cap Trans\cap Symm$ has crisp accessibility S, this defines a classical equivalence relation $\sim$ in W and thus $e(x,\Box\theta)=\inf_y\{Sxy\Rightarrow e(y,\theta)\}=\inf_{y\sim x}\{e(y,\theta)\}$ for any formula θ. Therefore, $e(x,\Box(\Box\varphi\vee\psi))=\inf_{y\sim x}\{\inf_{z\sim y}e(z,\varphi)\curlyvee e(y,\psi)\}$. But $\alpha_y=e(y,\Box\varphi)=\inf_{z\sim y}e(z,\varphi)$ is identical to α_x for all $y\sim x$ because $\{z:z\sim y\}=\{z:z\sim x\}$; hence, $e(x,\Box(\Box\varphi\vee\psi))=\inf_{y\sim x}\{\alpha_x\curlyvee e(y,\psi)\}=\alpha_x\curlyvee\inf_{y\sim x}\{e(y,\psi)\}=e(x,\Box\varphi\vee\Box\psi)$ by distributive properties of [0,1].

In the classical setting, S5 is characterized by Kripke models with *universal* accessibility relation; that is, $Sxy=1$ for all x,y. This cannot be the case for $\mathcal{G}\mathrm{S5}_{\Box\Diamond}$ or its $\Box$ -fragment due to the previous example, nor is it the case for the $\Diamond$ -fragment because $\neg\neg\Diamond\varphi\to\Diamond\neg\neg\varphi$ holds in all accessibility-crisp models but fails at the world v in the model displayed in the previous example ($\varphi:=q$). However,

THEOREM 5.2
$\mathcal{G}S5^*_{\Box\Diamond}:=\mathcal{G}S5_{\Box\Diamond}+\{\Box(\Box\varphi\vee\psi)\to(\Box\varphi\vee\Box\psi)\}$ is strongly complete for $\models_{GK\cap Universal}$; hence, for accessibility crisp models of $\mathcal{G}S5_{\Box\Diamond}$.

PROOF. Soundness follows from the above example. Weak completeness with respect to GK-models over universal frames is shown by Hájek in [19] for the deductively equivalent system $S5(G)$. This may be extended to strong completeness with respect to countable theories as in the proof of Theorem 3.1. ■

6 The algebraic connection

As an algebrizable deductive logic, $\mathcal{G}_{\Box\Diamond}$ has a unique algebraic semantics given by the variety of *bi-modal Gödel algebras*, those of the form $A=(G,I,K)$ where G is a Gödel algebra and I and K are unary operations in G satisfying the identities:

$$\begin{array}{ll}
I(a\cdot b)=Ia\cdot Ib & K(a\curlyvee b)=Ka\curlyvee Kb\\
I1=1 & K0=0\\
Ka\to Ib\leq I(a\to b) & K(a\to b)\leq Ia\to Kb
\end{array}$$

This means that $\mathcal{G}_{\Box\Diamond}$ is complete with respect to valuations $v:Var\to A$ in these algebras, when they are extended to $\mathcal{L}_{\Box\Diamond}$ interpreting $\Box$ and $\Diamond$ by I and K, respectively.

Similarly, $\mathcal{G}\mathrm{T}_{\Box\Diamond}$, $\mathcal{G}\mathrm{S4}_{\Box\Diamond}$ and $\mathcal{G}\mathrm{S5}_{\Box\Diamond}$ have for algebraic semantic the subvarieties of bi-modal Gödel algebras determined by the pairs of identities in the following table corresponding to their characteristic axioms:

$$\begin{array}{lll}
Ia\leq a & a\leq Ka & \text{reflexivity}\\
Ia\leq IIa & Ka\leq KKa & \text{transitivity}\\
a\leq IKa & KIa\leq a & \text{symmetry}\\
Ka\leq IKa & KIa\leq Ia & \text{euclidean property}
\end{array} \tag{6.1}$$

Notice that the algebraic models of $\mathcal{G}\mathrm{S4}_{\Box\Diamond}$ are just the bi-modal version of the topological pseudo-Boolean algebras of Ono [24] with a Gödel algebra as underlying Heyting algebra, and the algebraic models of $\mathcal{G}\mathrm{S5}_{\Box\Diamond}$ are the monadic Heyting algebras of Monteiro and Varsavsky [23] with a Gödel algebra as underlying Heyting algebra.

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Example. As we have noticed, there is no finite counter-model for the formula $\Box\neg\neg p \to \neg\neg\Box p$ in Gödel–Kripke semantics. However, the algebra $A = (\{0,a,1\}, I, K)$, where $\{0 < a < 1\}$ is the three elements Gödel algebra and $I1 = 1$, $Ia = I0 = 0$, $K1 = Ka = 1$, $K0 = 0$, is a bi-modal Gödel algebra (actually a monadic Heyting algebra) providing a finite counterexample to the validity of this formula by means of the valuation $v(p) = a$, as the reader may verify.

We may associate to each Gödel–Kripke frame $\mathcal{F} = (W,S)$ a bi-modal Gödel algebra $[0,1]^{\mathcal{F}} = ([0,1]^W, I^{\mathcal{F}}, K^{\mathcal{F}})$ where $[0,1]^W$ is the product Gödel algebra, and for each map $f \in [0,1]^W$:

$$I^{\mathcal{F}}(f)(w) = \inf_{w' \in W} (Sww' \Rightarrow f(w'))$$
$$K^{\mathcal{F}}(f)(w) = \sup_{w' \in W} (Sww' \cdot f(w'))$$

Theorem 6.1
$[0,1]^{\mathcal{F}}$ is a bi-modal Gödel algebra, and there is a one to one correspondence between Gödel–Kripke models over $\mathcal{F}$, and valuations $v: Var \to [0,1]^{\mathcal{F}}$ given by the adjunction:

$$\frac{Var \times W \xrightarrow{e} [0,1]}{Var \xrightarrow{v_e} [0,1]^W, \quad v_e(p) = e(-,p)}$$

so that $v_e(\varphi) = e(\_,\varphi)$ for any formula φ. Moreover, the transformation $\mathcal{F} \longmapsto [0,1]^{\mathcal{F}}$ sends reflexive, transitive, symmetric and euclidean Gödel–Kripke frames, respectively, into bi-modal algebras satisfying the corresponding identities.

Proof. The verification of the identities that $I^{\mathcal{F}}$, $K^{\mathcal{F}}$ must satisfy in each case is routine and the induction on formulas showing $v_e(\varphi)(w) = e(w,\varphi)$ is straightforward. ■

Reciprocally, utilizing our strong completeness theorem for normal theories (Theorem 3.2), we may associate to each countable bi-modal Gödel algebra A a GK-frame $\mathcal{F}$ such that A may be embedded in the associated algebra $[0,1]^{\mathcal{F}}$, and to each algebraic valuation v in A corresponds a GK-model over $\mathcal{F}$ validating the same formulas as v. However, the construction is not canonical.

Theorem 6.2
For any countable bi-modal Gödel algebra A there is Gödel frame $\mathcal{F} = (W,S)$ such that:
(i) Any pair of identities in (6.1) which is valid in A is valid in $[0,1]^{\mathcal{F}}$.
(ii) A is embeddable in the algebra $[0,1]^{\mathcal{F}}$.
(iii) For any valuation $v: Var \to A$ there exists $e_v: W \times Var \to [0,1]$ such that $(W,S,e_v) \models \varphi$ if and only if $v(\varphi) = 1$, for any sentence φ.

Proof. Fix a valuation η into A with onto extension $\eta: \mathcal{L}_{\Box\Diamond} \to A$ and let $T = \{\varphi : \eta(\varphi) = 1\}$. Then T is a normal theory deductively closed and for the model $M_T = (W,S,e)$ of Theorem 3.2 we have $M_T \models \varphi$ if and only if $\eta(\varphi) = 1$. Without loss of generality we may assume M_T is optimal (Lemma 4.2). Set $\mathcal{F} := (W,S)$, then (i) holds by Proposition 4.1 and the last claim of Theorem 6.1. To see (ii) notice that, by the same theorem, e induces a bi-modal Gödel valuation $v_e: Var \to [0,1]^{(W,S)}$, $v_e(p) = e(-,p)$ such that $v_e(\varphi) = e(-,\varphi) = \mathbf{1} \in [0,1]^W$ if and only if $\eta(\varphi) = 1$. This means that the extension $v_e: \mathcal{L}_{\Box\Diamond} \to [0,1]^{(W,S)}$ factors injectively through η; that is, $v_e = \delta \circ \eta$ for an injective homomorphism of bi-modal Gödel algebras $\delta: A \to [0,1]^{(W,S)}$. Finally, to show (iii), pick $v: Var \to A$, then $\delta \circ v$ is a valuation into $[0,1]^{(W,S)}$ which induces, by Theorem 6.1, a GK-valuation $e_v: W \times Var \to [0,1]$ such that $e_v(w,\varphi) = \delta(v(\varphi))(w)$. As δ is one to one we have that $v(\varphi) = 1$ if and only if $\delta(v(\varphi)) = \mathbf{1} \in [0,1]^W$; that is, $e_v(w,\varphi) = 1$ for all w, which means $(W,S,e_v) \models \varphi$. ■

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Applying parts (i) and (ii) of the previous theorem to the free algebras of countable rank we obtain:

COROLLARY 6.1
The variety of bi-modal Gödel algebras is generated by an algebra of the form $[0,1]^{(W,S)}$. A similar result holds for the subvarieties determined by any combination of identity pairs in Table (6.1).

As we defined $[0,1]^{(W,S)}$ we may define, similarly, bi-modal algebras $C^{(W,S)}$ where C is any complete chain, and obtain from Theorem 6.2, utilizing ultraproducts and the Dedekind–MacNeille completion:

THEOREM 6.3
For any bi-modal Gödel algebra A there is a complete chain C and a CK-frame (W,S) such that A is embeddable in $C^{(W,S)}$. Moreover, the latter algebra satisfies any identity pair in Table (6.1) satisfied by A.

7 Afterword

Our objective of axiomatizing the main bi-modal fuzzy logics under the Gödel–Kripke interpretation is fully achieved, and it is not difficult to extend this to languages enriched with sets of truth-constants along the lines of similar results for the uni-modal fragments in [6]. But some particular axiomatizability problems are left open in this article. We have emphasized already the lack of an axiomatization for the uni-modal fragments of $\mathcal{G}\mathrm{S5}_{\Box\Diamond}$. Another problem is the axiomatizability of validity in the accessibility-crisp models of each logic considered, having effective solutions for the fragments $\mathcal{G}_{\Box}$, $\mathcal{G}\mathrm{T}_{\Box}$ and $\mathcal{G}\mathrm{S4}_{\Box}$ (the logic themselves [6]), the fragment $\mathcal{G}_{\Diamond}$ (add the rule $(\varphi\to\psi)\vee\theta/(\Diamond\varphi\to\Diamond\psi)\vee\Diamond\theta$, Metcalfe and Olivetti [22]), and the logic $\mathcal{G}\mathrm{S5}_{\Box\Diamond}$ (the extension $\mathcal{G}\mathrm{S5}^*_{\Box\Diamond}$ introduced in Theorem 5.2).

The question of the decidability and complexity of $\mathcal{G}_{\Box\Diamond}$ and its extensions is also left unanswered since these logics do not have the finite model property under GK-semantics. However, the uni-modal fragments $\mathcal{G}_{\Box}$, $\mathcal{G}_{\Diamond}, \mathcal{G}\mathrm{T}_{\Diamond}$ and $\mathcal{G}\mathrm{S4}_{\Diamond}$ are known to be decidable, the first two by results of Metcalfe and Olivetti [22] who show they are PSPACE-complete, and the last three because they do have the finite model property (see [6]). As in [22], we may utilize a double negation interpretation of the classical modal logics into their Gödel counterparts to show that $\mathcal{G}\mathrm{S5}_{\Box\Diamond}$ is co-NP-hard and the other logics considered here are PSPACE-hard.

It has been noticed throughout the article that most results reported, excepting deductive completeness, hold for HK-models where H is an arbitrary complete Heyting algebra. It is reasonable to expect that validity in HK-models is axiomatized by $IK+L_H$, where IK is Heyting calculus plus the set of modal axioms of $\mathcal{G}_{\Box\Diamond}$ and L_H denotes an axiomatization of H-valued propositional logic. However, our completeness proof with respect to GK-models does not shed light on this hypothesis because it depends heavily on the linear and homogeneous character of [0,1].

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